Arrow's Impossibility Theorem

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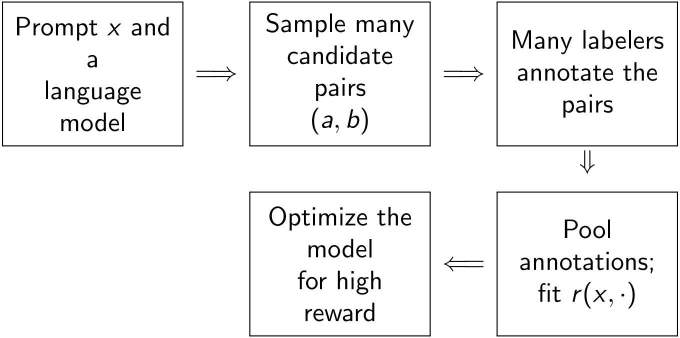
- Two requirements we might ask of an ideal aggregation rule.
- Arrow's theorem: these requirements force one person's ranking to determine the outcome.

Human Comparisons Become a Training Objective

Reinforcement learning from human feedback (RLHF) uses this pipeline:



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Whose ranking should it be?

A Running Example

One math prompt produces three responses:

Type	Primary concern	Ranking
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How might we combine these rankings into one?

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compare the number favoring a with the number favoring b.

But do the winners of the separate pairwise contests fit together into a single transitive ordering?

Coherent Individuals Can Produce a Cycle

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No scalar reward can represent all three comparisons:

$$r(a) > r(b) > r(c) > r(a).$$

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Matches the object learned by scalar RLHF better than it matches the goal of elections, its classical application.

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Axiom 1: Pareto Unanimity

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F satisfies *Pareto unanimity* if for every profile and every pair $a, b \in C$,

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It does not specify what to do when voters disagree.

Axiom 2: Independence of Irrelevant Alternatives

Definition

F satisfies *IIA* if the social comparison between a and b depends only on the voters' comparisons between a and b.
Formally, if profiles $\succ$ and $\succ'$ satisfy

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Voters may move other alternatives anywhere—only their a–b comparisons must remain fixed.

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Because both orders are complete, the social ranking exactly reproduces voter d's ranking.

A dictatorship satisfies Pareto and IIA—but ignores everyone except d.

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With m alternatives, each voter awards

$$m-1, m-2, \ldots, 0$$

points from first place to last place.

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Borda always produces a transitive score ranking—but other alternatives can affect the social comparison between a and b.

Borda Violates IIA

	Voter 1	Voter 2	Voter 3	Voter 4	Scores (a, b, c)
P	$a > c > b$	$a > c > b$	$b > a > c$	$b > a > c$	$(6, 4, 2)$

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- Every voter has the same a–b comparison at P and P'.
- Borda ranks a above b at P, but b above a at P'.
- Moving only the third alternative changes the social a–b comparison.

The Natural Rules Each Miss Something

Rule	Strict order?	Pareto?	IIA?
Dictatorship	✓	✓	✓

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- We saw pairwise majority cycle.
- We saw Borda reverse a and b after only c moved.
- Is there a fourth rule with all three checkmarks that is not a dictatorship?

Arrow's Impossibility Theorem

Theorem (Arrow)

Let $|C| \geq 3$. *Every social ranking rule*

$$F : \mathcal{L}(C)^n \longrightarrow \mathcal{L}(C)$$

defined on the unrestricted domain of strict preference profiles and satisfying Pareto unanimity and IIA is a dictatorship.

Proof Roadmap

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4. Use IIA to show that the same voter dictates every pair not involving b at every profile.

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4. Use IIA to show that the same voter dictates every pair not involving b at every profile.
5. Use a third alternative to extend control to pairs involving b.

Step 1: An Extreme Alternative Stays Extreme

Lemma

Suppose every voter places b either at the very top or at the very bottom of their ranking. Then the social order also places b at the top or at the bottom.

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We will modify the profile so that:

1. every comparison involving b remains unchanged;
2. every voter ranks c above a.

Proof of the Extreme-Alternative Lemma

Only the restriction to $\{a, b, c\}$ is shown:

$$\begin{array}{lccc} b \text{ at top} & b \succ_i a,\ b \succ_i c & \rightsquigarrow & b \succ'_i c \succ'_i a \\ b \text{ at bottom} & a \succ_i b,\ c \succ_i b & \rightsquigarrow & c \succ'_i a \succ'_i b \end{array}$$

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IIA preserves both social comparisons involving b:

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Pareto gives $c \succ'_F a$. But transitivity of $a \succ'_F b \succ'_F c$ gives $a \succ'_F c$—a contradiction.

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Pareto puts b socially at the bottom at $P^{(0)}$ and at the top at $P^{(n)}$. The extreme-alternative lemma says b is always socially top or bottom. So for some first index d, changing only voter d makes b jump from social bottom to social top.

What Have We Proved About Voter d?

Let
$$P^- := P^{(d-1)}, \qquad P^+ := P^{(d)}.$$

Then:

	voter d places b	society places b
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But this conclusion concerns only these two profiles.
Why must d control the social ranking at every other profile?

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To transfer this control to an arbitrary profile, we will construct an auxiliary profile that matches each pivotal profile on one relevant pair.

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3. Voters $i > d$ put b at the bottom.
4. Every voter's a–c comparison in Q matches their comparison in R.

Match Q to P^- and P^+ One Pair at a Time

At Q, the individual a–b comparisons match those at P^-:

	$i < d$	$i \geq d$
a vs. b	$b \succ_i a$	$a \succ_i b$

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The same voter d dictates every pair away from b, at every profile.

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Lemma

For every $a \neq b$ and every profile R,

$$a \succ_d b \iff a \succ_F^R b.$$

Step 4: Comparisons Involving b

We still need to show that voter d dictates comparisons of the form a versus b.

Lemma

For every $a \neq b$ and every profile R,

$$a \succ_d b \quad \Longleftrightarrow \quad a \succ_F^R b.$$

Choose a third alternative $c \notin \{a, b\}$.

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Pareto gives $b \succ_F^Q c$. The previous lemma gives $c \succ_F^Q a$.
Transitivity gives $b \succ_F^Q a$, and IIA transfers this comparison to R.

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Therefore, for every profile R and every pair a, c,

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The same fixed voter d controls every comparison at every profile. Thus F is a dictatorship. $\square$

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5. **Abandon IIA.** Borda and other score-based rules use the positions of other alternatives.

Thanks!

See you next class!